Name: ____________

Class: 12MTZ1

Teacher: MR KNOX

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2015 AP4

YEAR 12 TRIAL HSC EXAMINATION

MATHEMATICS EXTENSION 2

Time allowed - 3 HOURS
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES:

- Attempt all questions.
- Multiple choice questions are to be answered on the multiple choice sheet provided at the back of the paper.
- Each question in the extended response is to be commenced in a new booklet clearly marked Question 11, Question 12, etc on the top of the page and must show your name and class.
- All necessary working should be shown in every question for extended response. Full marks may not be awarded for careless or badly arranged work.
- Board of Studies approved calculators may be used. Standard Integral Tables are provided.
- Write your name and class in the space provided at the top of this question paper.

1) Which pair of equations gives the directrices of $y = \frac{4}{x}$

(A) $y = \pm(x + 2\sqrt{2})$

(B) $y = -x \pm 4\sqrt{2}$

(C) $y = \pm(x + 4\sqrt{2})$

(D) $y = -x \pm 2\sqrt{2}$

2) Which expression is equal to $\int \sec x dx$

(A) $\sec x \tan x + C$

(B) $\tan x + C$

(C) $\ln|\sec x + \tan x| + C$

(D) $\ln|\sin x| + C$

3) For a complex number $x + yi = \sqrt{2 + 3i}$

(A) $x^2 + y^2 = 3 \text{ and } 2xy = 2$

(B) $x^2 + y^2 = 2 \text{ and } -2xy = 3$

(C) $x^2 - y^2 = 2 \text{ and } 2xy = 3$

(D) $x^2 + y^2 = 2 \text{ and } 2xy = 3$

4) For the relation $x^2 + e^y = 7$

(A) $\frac{dy}{dx}(2x + e^y) = 0$

(B) $\frac{dy}{dx} = -\frac{2x}{e^y}$

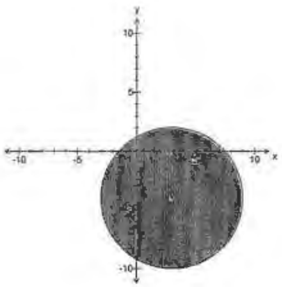
(C) $\frac{dy}{dx} = 2x$

(D) $\frac{dy}{dx}2x + \frac{dx}{dy}e^y = 0$

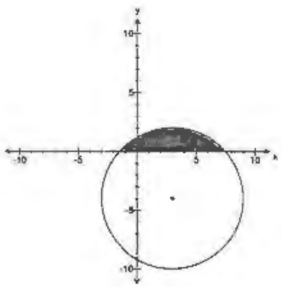
5) Which region on the argand diagram shows the complex number z, which is defined by $|z - 3 + 4i| \leq 6$ *and* $Im(z) \geq 0$?

The centre of each circle is shown for convenience

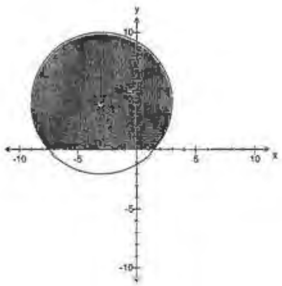
(A)



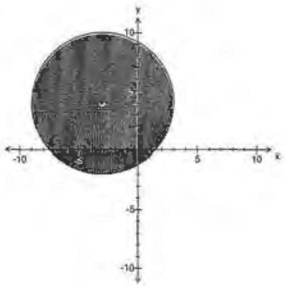
(B)



(C)



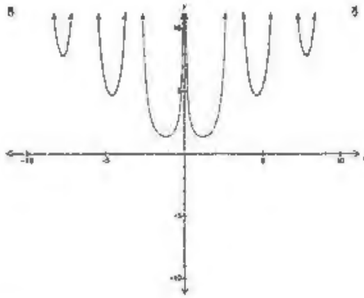
(D)



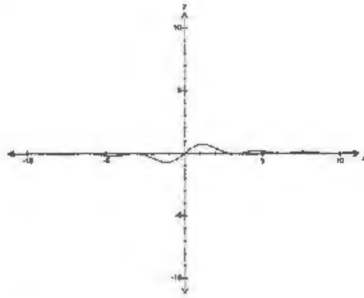
6) Which diagram best represents the graph

$$y = \frac{x}{\sin^2 x}$$

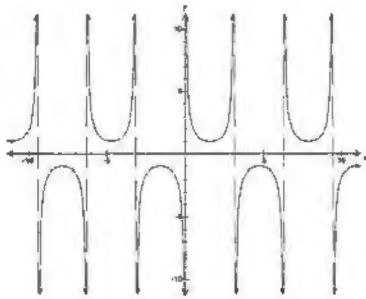
(A)



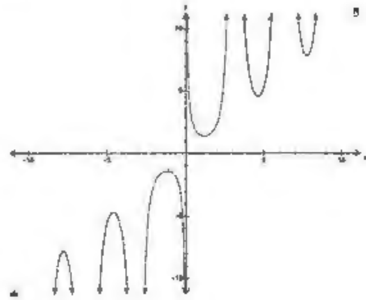
(B)



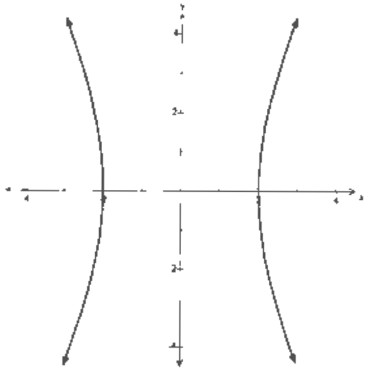
(C)



(D)



Questions 7 and 8 refer to the graph of $\frac{x^2}{4} - \frac{y^2}{16} = 1$



7) What is the eccentricity?

(A) $\sqrt{2}$

(B) $\sqrt{\frac{3}{2}}$

(C) $\frac{\sqrt{5}}{2}$

(D) $\sqrt{5}$

8) What are the equations of the asymptotes?

(A) $y = \pm 2x$

(B) $y = \pm \frac{1}{2}x$

(C) $y = \pm 2$

(D) $y = \pm \frac{1}{2}$

9) The equation $x^4 + px + q = 0$, where $p \neq 0$ and $q \neq 0$, has roots α, β, γ and δ.

What is the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$?

(A) $-4q$

(B) $p^2 - 2q$

(C) $p^4 - 2q$

(D) p^4

10) What is the value of $S = \sum_{r=1}^{\infty} rp(1-p)^{r-1}$

(A) $S = 1$

(B) $S = \frac{1}{p}$

(C) $S = \frac{1}{1-p}$

(D) $S = \frac{1}{p(1-p)}$

Question 11 (15 Marks)

a) Let $z = 1 + i\sqrt{3}$ and $w = -\sqrt{3} + i$

(i) Express w in terms of z. 1

(ii) Express z in modulus-argument form. 2

(iii) Write z^{10} in the form $a + ib$ 2

b) Find numbers A, B and C such that

$\frac{6x^2+9x+4}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$ 2

c) Find the value of z if $z^2 + 3iz = 7$ 2

d Evaluate $\int_0^{1/2} \frac{x}{\sqrt{1-x^2}} dx$ 3

e) Sketch on the argand diagram the region defined by 3

$\frac{\pi}{4} < Re\,(z) < \frac{\pi}{2}$ AND $\frac{\pi}{4} < Im\,(z) < \frac{\pi}{2}$ AND $\frac{\pi}{4} < arg\,(z) < \frac{\pi}{2}$

Question 12 (15 Marks)

a) (i) Show that for $t = \tan\frac{x}{2}$, 1

$$dx = \frac{2dt}{1+t^2}$$

(ii) Use the substitution $t = \tan\frac{x}{2}$, to evaluate 3

$$\int_0^{\pi/2} \frac{1}{1 + cosx + sinx} dx$$

b) By using an appropriate substitution , evaluate $\int_3^8 \frac{x}{(x+1)-\sqrt{x+1}} dx$ 4

Hint: Consider a substitution which removes the $\sqrt{}$

c) Sketch the graph of $y = \frac{x^3+3x^2+x+4}{x+3}$, showing all asymptotes 3

d) Find and sketch the locus of the complex number z, where 4

$$3|z - (2 + 2i)| = |z - (6 + 6i)|$$

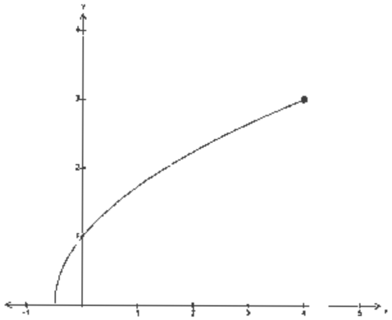
Question 13 (15 Marks)

(a) i) Given $I_n = \int x(x^5+1)^n dx$, show that 3

$$I_n = \frac{x^2(x^5+1)^n}{2+5n} + \frac{5n}{2+5n} I_{n-1}$$

ii) Hence, find I_{10} in terms of I_8 and x. 2

(b) The following graph of $y = f(x)$ starts at $x\ intercept = -\frac{1}{2}$ and ends at the point (4,3). Sketch



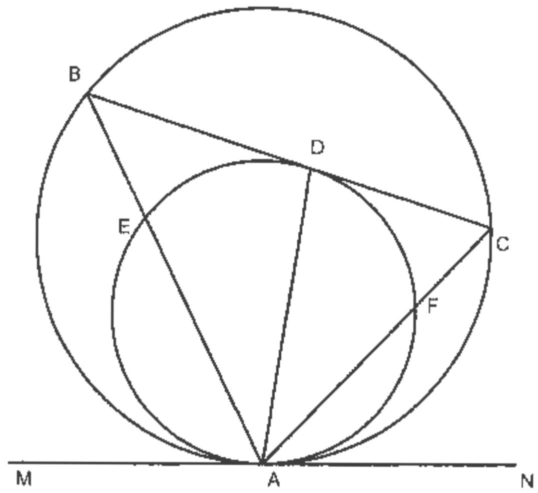
(i) $y = \frac{1}{f(x)}$ 1

(ii) $y = f(x-1)+1$ 1

(iii) $y = \log(f(x))$ 2

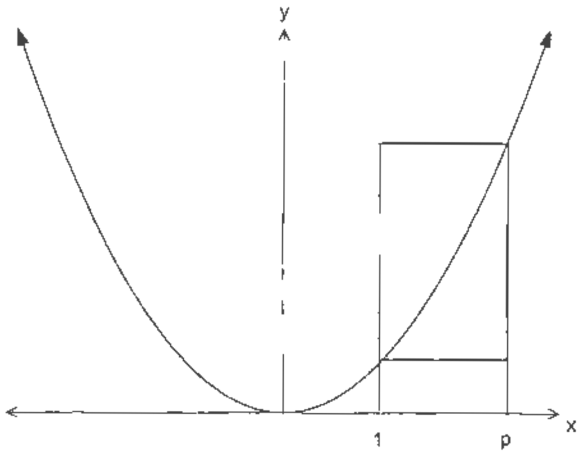
(iv) $y = f(x^2)$ 2

(c) In the diagram, MAN is the common tangent of two circles touching internally at A. 4
B and C are two points on the larger circle such that BC is a tangent to the smaller circle with point of contact D. AB and AC cut the smaller circle at E and F respectively. Show that AD bisects $\angle BAC$



Question 14 (15 Marks)

a) Copy the graph of $y = x^2$ into your writing booklet. 3



By considering this graph, show that $p - 1 < \frac{p^3}{3} - \frac{1}{3} < p^3 - p^2$ for $p > 1$

b) In a series $T_1 = 3, T_2 = 8$ and $T_n = 2T_{n-1} - T_{n-2}$ for $n > 2$. Show that $T_n = 5n - 2$ for $n \geq 1$. 3

c) The polynomial $P(x) = x^6 + ax^3 + bx^2$ has a factor $(x+1)^2$. Find the value of the real numbers a and b. 3

d) The equation $x^4 + bx^3 + cx^2 + dx + k = 0$ has roots $\alpha, \frac{1}{\alpha}, \beta$ and $\frac{1}{\beta}$.

(i) Show that $k = 1$ 1

(ii) Show $b = d$ 2

e) The equation $x^3 + 3x^2 + 2x + 1 = 0$ has roots α, β and γ. Find the monic cubic equation with roots α^2, β^2 and γ^2. 3

Question 15 (15 Marks)

a) A rectangular hyperbola has the equation $xy = \frac{1}{2}a^2$.

(i) Show that the point $P\left(\frac{at}{2}, \frac{a}{t}\right)$ lies on the hyperbola, for all values of t. 1

(ii) Show that the equation of the tangent at P is given by the equation $2x + t^2y = 2at$. 2

(iii) One of the foci of the hyperbola is $S(a, a)$. DO NOT PROVE THIS. From a point T on the tangent a line is drawn through S, such that PS is perpendicular to PT. Show that the equation of PS is 2

$$t^2x - 2y = at^2 - 2a$$

(iv) Hence find the locus of the point T. 2

b) (i) By starting with $\left(\cos\frac{x}{2} + i\sin\frac{x}{2}\right)^2$ and using De Moivre's Theorem find expressions for $\sin x$ and $\cos x$ in terms of $\sin\frac{x}{2}$ and $\cos\frac{x}{2}$. 2

(ii) By using the results from part (i) and $t = \tan\frac{x}{2}$, prove the result 1

$$\tan x = \frac{2t}{1 - t^2}$$

c) Let $z = \cos\theta + i\sin\theta$ and let $\omega = \cos\alpha + i\sin\alpha$

(i) Solve $z^7 = 1$ 1

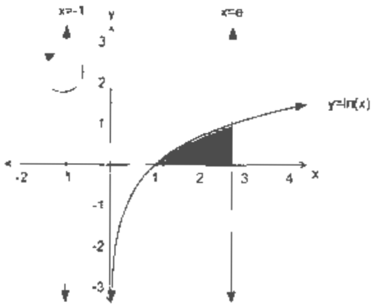
(ii) Show that $\cos n\alpha = \frac{1}{2}(\omega^n + \omega^{-n})$ 1

(iii) Suppose that $\frac{1}{2} + \cos\alpha + \cos 2\alpha + \cos 3\alpha = 0$. Use part (ii) to show that $\omega^7 = 1$. 2

(iv) Hence solve $\frac{1}{2} + \cos\alpha + \cos 2\alpha + \cos 3\alpha = 0$ for $\pi < \alpha \leq \pi$ 1

Question 16 (15 Marks)

a) The diagram shows the area bounded by the function $y = \ln(x)$, the x-axis and the line $x = e$. 4



Find the volume formed when the area is rotated around the line $x = -1$.

b) A $20kg$ trolley is pushed with a force of $100N$. Friction causes a resistive force which is proportional to the square of the trolley's velocity.

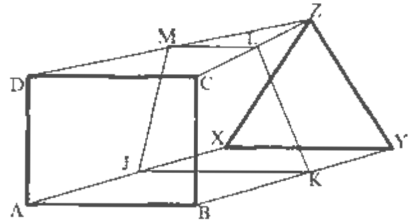
(i) Show that $\ddot{x} = 5 - \frac{kv^2}{20}$, where k is a positive constant. 1

(ii) If the trolley is initially stationary at the origin, show that the distance travelled when its speed is V is given by 4

$$x = \frac{10}{k}\ln\left(\frac{100}{100 - kV^2}\right)$$

QUESTION 16 CONTINUES NEXT PAGE

c) In the diagram below, a solid is shown which has a rectangular front which is parallel to the triangular back and the distance between the front and back is $30cm$.



Also $AB = DC = XY = 20cm$, $DA = 8cm$ and the perpendicular distance from Z to XY is also $8cm$.

(i) Copy the diagram into your writing booklet and add suitable coordinate axes to your diagram. 1

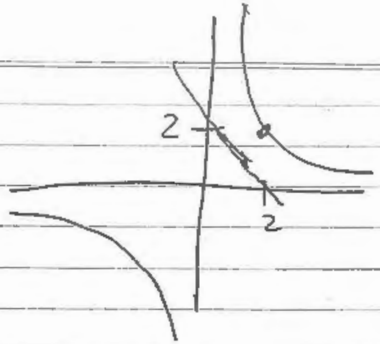
(ii) Find expressions for the length of JK and ML 2

(iii) Find the area of the cross section $JKLM$ 1

(iv) Hence find the volume of the solid. 2

END OF PAPER

① $y = \frac{4}{x}$



$e = \sqrt{2}$

directrix passes through $\left(\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}\right)$ Ⓓ

$= (\sqrt{2}, \sqrt{2})$

$m = -1$

$y - \sqrt{2} = -1(x - \sqrt{2})$

$y = -x + \sqrt{2} + \sqrt{2}$

$y = -x + 2\sqrt{2}$

other is $y + \sqrt{2} = -1(x + \sqrt{2})$

$y = -x - 2\sqrt{2}$

Incorrect Ⓐ - places $\pm$ at front.

Incorrect Ⓑ - Wrong point on directrix used.

ie $(2\sqrt{2}, 2\sqrt{2})$

$y - 2\sqrt{2} = -(x - 2\sqrt{2})$

$y = -x + 4\sqrt{2}$

Incorrect Ⓒ $y = \pm(x + 4\sqrt{2})$

error Ⓐ and Ⓑ

② Ⓒ Correct.

$$\int \sec x \, dx = \int \sec x \, \frac{\sec x + \tan x}{\sec x + \tan x} \, dx$$

$$\left[\begin{array}{l} \text{let } u = \sec x + \tan x \\ du = (\sec x \tan x + \sec^2 x) dx \end{array}\right]$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$$= \int \frac{1}{u} \, du$$

$$= \ln|u|$$

$$= \ln|\sec x + \tan x)$$

Ⓐ Ⓑ incorrect std Integral.

Ⓓ incorrect

Wrong $\int \sec x \, dx = \int (\cos x)^{-1} dx$

$= \ln(\sin x)$ Wrong.

(3) $x+iy=\sqrt{2+3i}$

$(x+iy)^2 = 2+3i$

$x^2+2ixy-y^2=2+3i$ (C)

$x^2-y^2=2 \qquad 2xy=3$

(4) $x^2+e^y=7$

$2x+\frac{dy}{dx}e^y=0$

$\frac{dy}{dx}e^y=0-2x$

$\frac{dy}{dx}=\frac{-2x}{e^y}$ (B)

(5) Circle is centred at $(3,-4)$
radius of 6
$\mathrm{Im}(z)\geq 0 \longrightarrow$ above x-axis. (B)

(6) $\frac{\text{Top}}{\text{Bottom}}=\frac{-}{+}=- \quad \Big| \quad \frac{\text{Top}}{\text{Bottom}}=\frac{+}{+}=1$

undefined when $\sin x=0$ ie $x=0,\pi,2\pi$

(D)

(7) $b^2=a^2(e^2-1)$

$16=4(e^2-1)$

$4=e^2-1$ (D)

$5=e^2$

$e=\sqrt{5}$

(8) $y=\pm\frac{b}{a}x \longrightarrow y=\pm 2x$ (A)

(9)
$$\left.\begin{aligned} \alpha^4+p\alpha+q&=0\\ \beta^4+p\beta+q&=0\\ \gamma^4+p\gamma+q&=0\\ \delta^4+p\delta+q&=0 \end{aligned}\right\}$$

$\alpha^4+\beta^4+\gamma^4+\delta^4=-p(\alpha+\beta+\gamma+\delta)-4q$

by $\alpha+\beta+\gamma+\delta=\frac{-b}{a}$
$=0$

$\alpha^4+\beta^4+\gamma^4+\delta^4=-4q$

(A)

(10) $S=p+2p(1-p)+3p(1-p)^2+ \ldots$ (1)

$(1-p)S=p(1-p)+2p(1-p)^2+3p(1-p)^3+ \ldots$ (2)

(1) − (2)

$pS=p+p(1-p)+p(1-p)^2+ \ldots$

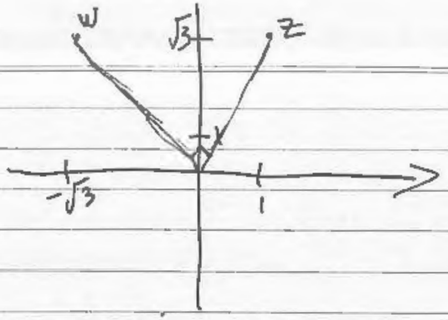
$S=1+(1-p)+(1-p)^2+ \ldots$

$a=1 \quad r=1-p$

$S_\infty=\frac{1}{1-(1-p)}$

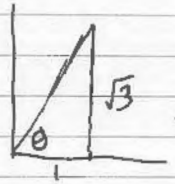
$=\frac{1}{p}$ (B)

11 (a) (i)



$w = zi$

(ii)



$|z| = \sqrt{\sqrt{3}^2 + 1^2}$

$= 2$

$\tan(\arg z) = \sqrt{3}/1$

$\arg z = \pi/3$

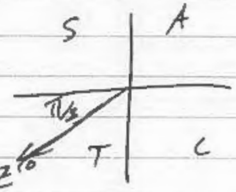
$z = 2(\cos \pi/3 + i \sin \pi/3)$

(iii) $z^{10} = 2^{10}(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})^{10}$

$= 1024(\cos \frac{\pi}{3} \times 10 + i \sin \frac{\pi}{3} \times 10)$

$= -1024 \times \frac{1}{2} + i(-1024 \times \frac{\sqrt{3}}{2})$

$= -512 - 512\sqrt{3}i$



11

(b) $\frac{6x^2+9x+4}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$6x^2+9x+4 = A(x^2+1) + (Bx+C)x$

$x^2: \quad 6 = A + B \quad (1)$

$x: \quad 9 = C \quad (2)$

const: $A = 1 \quad (3)$

from (1) & (3)

$B = 5$

$A = 1, B = 5, C = 9$

(c) $z^2 + 3iz = 7$

$z^2 + 3iz - 7 = 0$

$z = \frac{-3i \pm \sqrt{-9+28}}{2}$

$= \frac{-3i}{2} \pm \sqrt{14}$

$= \pm\sqrt{14} - \frac{3i}{2}$

(d)

(d) $\displaystyle\int_{\cancel{2}\,0}^{\cancel{3}\,1/2} \frac{x}{\sqrt{1-x^2}}\,dx$

let $u = 1 - x^2$

$du = -2x\,dx \rightarrow x\,dx = \dfrac{du}{-2}$

when $x = \cancel{2}\,0$

$u = 1 - \cancel{2^2}$

$= \cancel{-3}\ 1$

when $x = \cancel{3}\ 1/2$

$u = 1 - \cancel{3^2}\ \ 1 - {1/2}^2$

$= \cancel{-8}\ \ 3/4$

$$= \int_{\cancel{-3}\,1}^{\cancel{-8}\,3/4} \frac{1}{\sqrt{u}}\,\frac{du}{-2}$$

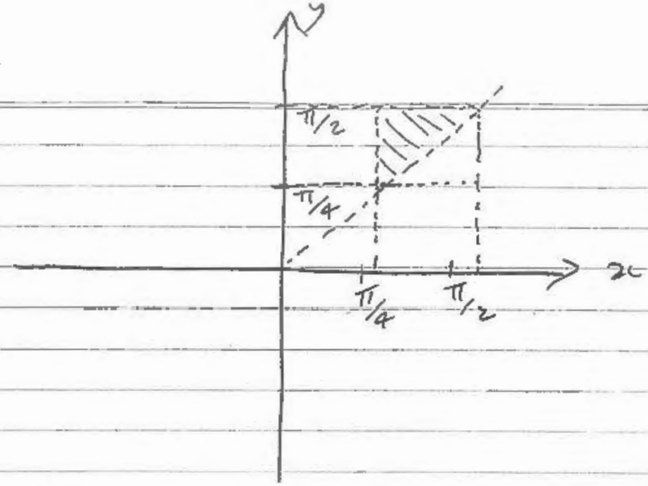
$$= \frac{1}{-2}\int_{\cancel{-3}\,1}^{\cancel{-8}\,3/4} u^{-1/2}\,du$$

$$= \left[2u^{1/2}\right]_1^{3/4}$$

$$= 2\sqrt{3/4} - 2$$

$$= \sqrt{3} - 2$$

(e)



12

(a)(i) $t = \tan \frac{x}{2}$

$\tan^{-1} t = \frac{x}{2}$

$2\tan^{-1} t = x$

$\frac{dx}{dt} = 2\frac{1}{1+t^2}$

$dx = \frac{2dt}{1+t^2}$

(ii) ~~when $x = \pi/2$~~

~~$t = \tan \frac{\pi}{4}$~~

~~$= 1$~~

~~$x = 0$~~

~~$t = \tan 0$~~

~~$= 0$~~

✓

~~$= \int_0^1 \frac{1}{\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2}} \frac{2dt}{1+t^2}$~~ ✓

~~$= \int_0^1 \frac{1}{\frac{1-t^2+2t}{1+t^2}} \frac{2dt}{1+t^2}$~~

~~$= 2\int_0^1 \frac{1}{1+2t-t^2} dt$~~

~~$=$~~

12 (a) (ii)

$$\int_0^{\pi/2} \frac{1}{1+\cos x + \sin x} dx$$

$x = \pi/2 \quad t = \tan \pi/4$

$= 1$

$x = 0 \quad t = \tan 0$

$= 0$

$$= \int_0^1 \frac{1}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \frac{2dt}{1+t^2}$$ ✓

$$= \int_0^1 \frac{2(1+t^2)}{1+t^2+2t+1-t^2} dt$$

$$= \int_0^1 \frac{1}{t+1} dt$$ ✓

$$= \Big[\ln(t+1)\Big]_0^1$$

$= \ln 2 - \ln 1$

$= \ln 2$ ✓

(12) (b)

$$\int_3^8 \frac{x}{(x+1)-\sqrt{x+1}}\,dx$$

let $u^2 = x+1$
$x = u^2 - 1$
$dx = 2u\,du$ ✓

when $x = 3$, $u^2 = 3+1$, $u = 2$
$x = 8$, $u^2 = 8+1$, $u = 3$ ✓

$$= \int_2^3 \frac{u^2-1}{u^2-u}\,2u\,du$$

$$= \int_2^3 \frac{(u+1)(u-1)}{u(u-1)}\,2u\,du \quad ✓$$

$$= \int_2^3 2(u+1)\,du \quad ✓ \text{ full here}$$

$$= 2\left[\frac{u^2}{2}+u\right]_2^3$$

$$= 2\left[\frac{9}{2}+3-2-2\right]$$

$=$

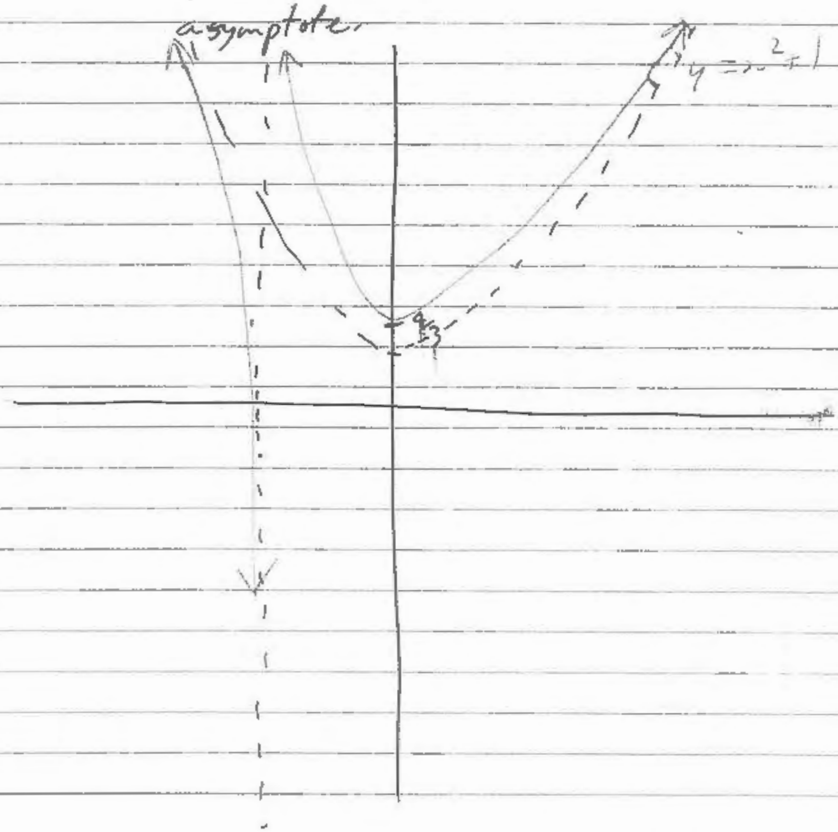
(c) $y = \dfrac{x^3+3x^2+x+4}{x+3}$

$$\begin{array}{r} x^2 \quad + 1 \\ x+3 \,\big)\, x^3+3x^2+x+4 \\ x^3+3x^2 \qquad\quad \\ \hline x+4 \\ x+3 \\ \hline 1 \end{array}$$

y int
$x = 0$
$y = 1 + \frac{1}{3}$
$= \frac{4}{3}$

$y = x^2+1 + \dfrac{1}{x+3}$

↑
asymptote



(d) let $z = x+iy$

$$3\left|x+iy-(2+2i)\right| = \left|x+iy-(6+6i\right|$$

$$3\sqrt{(x-2)^2+(y-2)^2} = \sqrt{(x-6)^2+(y-6)^2}$$

$$3^2(x^2-4x+4+y^2-4y+4) = x^2-12x+36+y^2-12y+36$$

$$9x^2-36x+36+9y^2-36y+36 = x^2-12x+\cancel{36}+y^2-12y+\cancel{36}$$

$$8x^2-24x+9y^2-24y=0$$

$$x^2-3x+y^2-3y=0$$

$$x^2-3x+\left(-\tfrac{3}{2}\right)^2+y^2-3y+\left(-\tfrac{3}{2}\right)^2 = 2\times\left(\tfrac{3}{2}\right)^2$$

$$\left(x-\tfrac{3}{2}\right)^2+\left(y-\tfrac{3}{2}\right)^2 = \tfrac{9}{2}$$

$$\left(x-\tfrac{3}{2}\right)^2+\left(y-\tfrac{3}{2}\right)^2 = \left(\tfrac{3}{\sqrt{2}}\right)^2$$

sketch.

(13) (a) $I_n = \int x(x^5+1)^n\,dx$

$v = (x^5+1)^n \qquad u' = x$

$v' = 5x^4 n(x^5+1)^{n-1} \qquad u = \frac{x^2}{2}$

$$I_n = \frac{x^2}{2}(x^5+1)^n - \int \frac{x^2}{2}5x^4 n(x^5+1)^{n-1}$$

$$= \frac{x^2}{2}(x^5+1)^n - \frac{5n}{2}\int x^6(x^5+1)^{n-1}$$

$$= \frac{x^2}{2}(x^5+1)^n - \frac{5n}{2}\int x\cdot x^5(x^5+1)^{n-1}$$

$$= \frac{x^2}{2}(x^5+1)^n - \frac{5n}{2}\int x\left[(x^5+1)-1\right](x^5+1)^{n-1}\,dx$$

$$= \frac{x^2}{2}(x^5+1)^n - \frac{5n}{2}\int x(x^5+1)^n - x(x^5+1)^{n-1}\,dx$$

$$I_n = \frac{x^2}{2}(x^5+1)^n - \frac{5n}{2}I_n + \frac{5n}{2}I_{n-1}$$

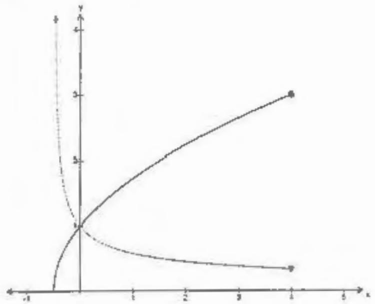
$$\left(\frac{5n}{2}+1\right)I_n = \frac{x^2}{2}(x^5+1)^n + \frac{5n}{2}I_{n-1}$$

$$\frac{5n+2}{2}I_n = \frac{x^2}{2}(x^5+1)^n + \frac{5n}{2}I_{n-1}$$

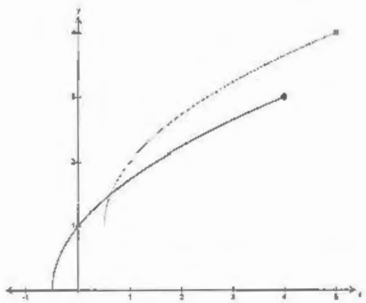
$$I_n = \frac{x^2(x^5+1)^n}{5n+2} + \frac{5n}{5n+2}I_{n-1}$$

ANSWERS 13(b)

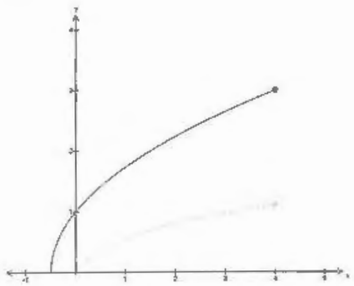
(i)



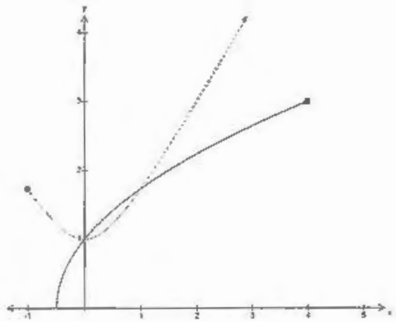
(ii)



(iii)



(iv)



13 (a) (ii)

$$I_{10} = \frac{x^2(x^5+1)^{10}}{52} + \frac{50}{52} I_9 \checkmark$$

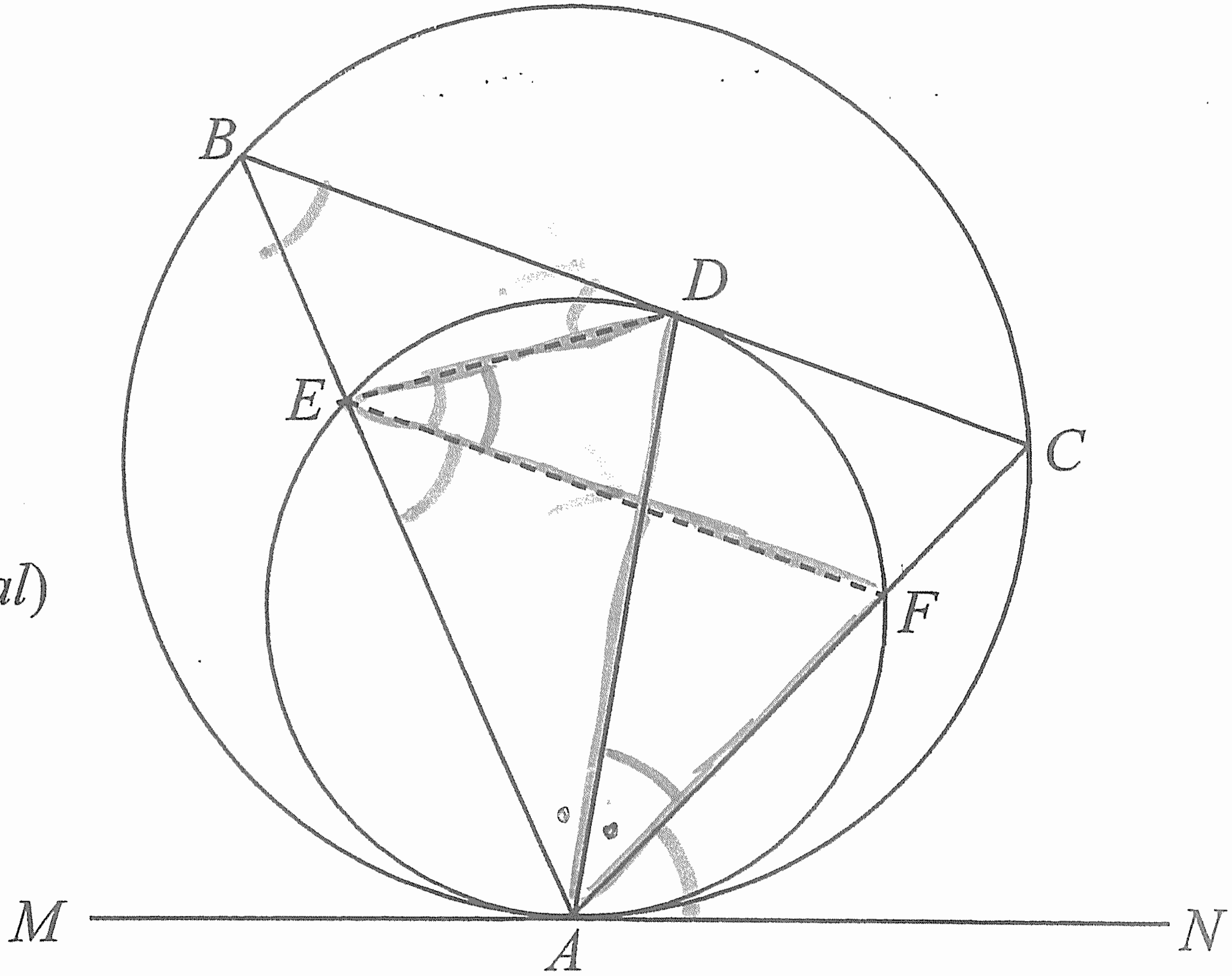
$$= \frac{x^2(x^5+1)^{10}}{52} + \frac{50}{52}\left[\frac{x^2(x^5+1)^9}{47} + \frac{45}{47} I_8\right] \checkmark$$

13 (b) see printed graphs

es *DEF* and *BDE*	1
es *DEF* and *DAF*	1
es *BDE* and *DAE*	1

ıd *ED*.
ıent theorem in

∠'s on transversal AB)
's within parallel lines
ual)
ɔtended by same arc DF at
rence of circle AEF are equal)
nent theorem in circle AEF)



- applies alternate segment theorem to deduce *EF* and *BC* are parallel
- deduces equality of angles *DEF* and *BDE*
- deduces equality of angles *DEF* and *DAF*
- deduces equality of angles *BDE* and *DAE*

Answer

Construct *EF* and *ED*.

Applying the alternate segment theorem in circles ABC and AEF :

$\angle ABC = \angle NAC = \angle AEF$

$\therefore EF \parallel BC$ (*equal corresp. $\angle$'s on transversal AB*)

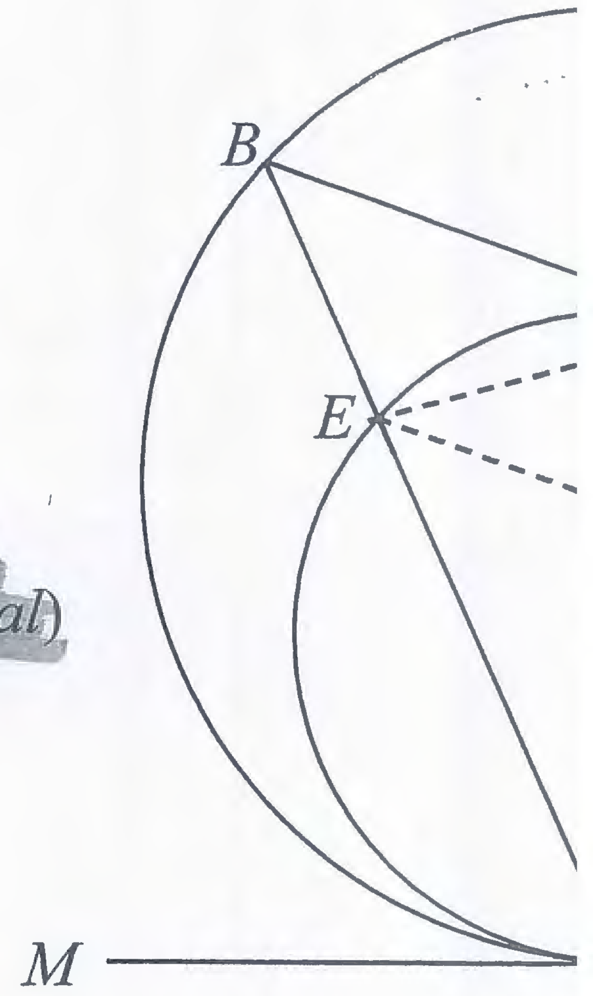
$\therefore \angle DEF = \angle BDE$ (*Alt. $\angle$'s within parallel lines are equal*)

2 $\angle DEF = \angle DAF$ (*$\angle$'s subtended by same arc DF at circumference of circle AEF are equal*)

$\angle BDE = \angle DAE$ (*Alt. segment theorem in circle AEF*)

$\therefore \angle DAF = \angle DAE$ $\quad \angle DAF = \angle DEF = \angle BDE = \angle DAE$

Hence *AD* bisects $\angle BAC$



14 (a)

lower < $\int$ < upper.

$1^2(p-1) < \int_1^p x^2\,dx < p^2(p-\cancel{1})$ ✓

$p-1 < \left[\frac{x^3}{3}\right]_1^p < p^3-p^2$ ✓

$p-1 < \frac{p^3}{3} - \frac{1}{3} < p^3 - p^2$ ✓

(b) For $n=1$

$T_1 = 5\times1-2=3$ as required.

$T_2 = 5\times2-2 =$ " "

true for $n=1,2$

assume true for $n=k-1$ $n=k-2$

ie ~~$T_k = 2T_{n-1}$~~

$T_{k-1} = 5(k-1)-2$ $T_{k-2}=5(k-2)-2.$

for $n=k$ RTP $T_k = 5k-2.$

$T_k = 2T_{k-1} - T_{k-2}$

$= 2(5(k-1)-2) - \cancel{(5(k-2)} - (5(k-2)-2)$

$= 10k - 5k \underbrace{-10-4+10}_{\text{cancel.}} +2$

$= 5k-2$

as required.

14 (c) $P(x) = x^6 + ax^3 + bx^2$

$P(-1) = 1 - a + b = 0$ (1) ✓

$P'(x) = 6x^5 + 3ax^2 + 2bx$

$P'(-1) = -6 + 3a - 2b = 0$ (2) ✓

from (1)

$a = 1+b$

sub into (2)

$-6+3(1+b)-2b=0$

$-3+3b-2b=0$

$b=3$

$a = 1+3$ ✓

(d) (i) Product $= e/a$

$k = \alpha \frac{1}{\alpha} \beta \frac{1}{\beta}$ ✓

$= 1$

(ii) α ~~for example~~

~~satisfys~~

sub in α.

$\alpha^4 + b\alpha^3 + c\alpha^2 + d\alpha + 1 = 0$

sub ~~in $1/\alpha$~~

(c) $x^3+3x^2+2x+1=0$

$\alpha^2, \beta^2, \gamma^2$

let $X=x^2$

$\sqrt{X}=x$

$x=X^{1/2}$

$(X^{1/2})^3+3(X^{1/2})^2+2(X^{1/2})+1=0$ ✓

$X^{1/2}(X+2)=-(1+3X)$ ✓

square

$X(X^2+4X+4)=1+6X+9X^2$

$X^3+4X^2+4X=1+6X+9X^2$

$\cancel{X^3-5X^2+10X-1=0}$

$X^3-5X^2-2X-1=0$ ✓

(13) (a) (i) $x=\frac{at}{2}$ $\quad y=\frac{a}{t}$

$\frac{2x}{a}=t$ $\quad t=\frac{a}{y}$

$\frac{2x}{a}=\frac{a}{y}$ ✓

$xy=\frac{a^2}{2}$

(ii) $\frac{dx}{dt}=\frac{a}{2}$ $\quad y=at^{-1}$

$\frac{dy}{dt}=\cancel{at}\ -1a\,t^{-2}$

$=\frac{-a}{t^2}$

$\frac{dy}{dx}=-a/t^2\div a/2$

$=\frac{-a}{t^2}\times\frac{2}{a}$

$=-2/t^2$ ✓

eqn $y-\frac{a}{t}=\frac{-2}{t^2}\left(x-\frac{at}{2}\right)$

$t^2y-at=-2x+at$ ✓

$2x+t^2y=2at$

(iii)

$m_{tangent} \times m_{\perp} = -1$

$-\frac{2}{t^2} \times m_{\perp} = -1$

$m_{\perp} = \frac{t^2}{2}$ ✓

$y - a = \frac{t^2}{2}(x - a)$

$2y - 2a = t^2x - at^2$ ✓

$at^2 - 2a = t^2x - 2y$

$t^2x - 2y = at^2 - 2a$

(iv) T lies on the 2 lines from (ii) & (iii)

$$\begin{cases} 2x + t^2y = 2at & (1) \\ t^2x - 2y = at^2 - 2a. & (2) \end{cases}$$

Square (1) & (2)

$(1)^2 \quad (2x)^2 + 2 \times 2x\,t^2y + t^4y^2 = 4a^2t^2$ ✓

$4x^2 + 4xt^2y + t^4y^2 = 4a^2t^2 \quad (3)$

$(2)^2 \quad t^4x^2 - 4t^2xy + 4y^2 = a^2t^4 - 4a^2t^2 + 4a^2 \quad (4)$

cancel cancel

$(3)+(4) \quad 4x^2 + t^4x^2 + 4y^2 + t^4y^2 = \cancel{4a^2 + 4a^2t^2}$

$= a^2t^4 + 4a^2$

$x^2(4+t^4) + y^2(4+t^4) = a^2(t^4+4)$

$\div\ 4+t^4$

$x^2 + y^2 = a^2$ ✓

(b) (i) $\left(\cos\frac{x}{2} + i\sin\frac{x}{2}\right)^2$

$= \cos\frac{2x}{2} + i\sin\frac{2x}{2}$

$= \cos x + i\sin x$

Also

$\left(\cos\frac{x}{2} + i\sin\frac{x}{2}\right)^2 = \cos^2\frac{x}{2} + 2i\sin\frac{x}{2}\cos\frac{x}{2} - \sin^2\frac{x}{2}$ ✓

Real $\quad \cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$ ✓

Im $\quad \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$

(ii) ~~$t = \tan\frac{x}{2}$ $\tan x = \frac{\sin x}{\cos x}$~~

~~$= \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$~~

(ii) $\tan x = \frac{\sin x}{\cos x}$

$= \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}$

$\div \cos^2\frac{x}{2} \quad = \frac{2\tan\frac{x}{2}}{1 - \tan^2\frac{x}{2}}$ ✓

$= \frac{2t}{1-t^2}$

(8)(i) $z^7=1$

$z = 1, \text{cis}\,\frac{2\pi n}{7}$

$n=\pm1,\pm2,\pm3$

OR

$z=\text{cis}\,\frac{2\pi n}{7} \quad n=0,\pm1,\pm2,\pm3$

(ii) $\frac{1}{2}(\text{cis}\,n\theta+\text{cis}(-n\theta))$

$=\frac{1}{2}(\cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta)$ ✓

$=\frac{1}{2}\times 2\cos n\theta$

$=\cos n\theta$

$\frac{1}{2}+\cos\alpha+\cos 2\alpha+\cos 3\alpha$

(iii) $\frac{1}{2}(1+\omega+\omega^{-1}+\omega^2+\omega^{-2}+\omega^3+\omega^{-3})=0$

$\times 2$

$(1+\omega+\omega^{-1}+\omega^2+\omega^{-2}+\omega^3+\omega^{-3})=0$

$\omega^{-3}+\omega^{-2}+\omega^{-1}+1+\omega+\omega^2+\omega^3=0$

$\times\omega^3 \quad 1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6=0$

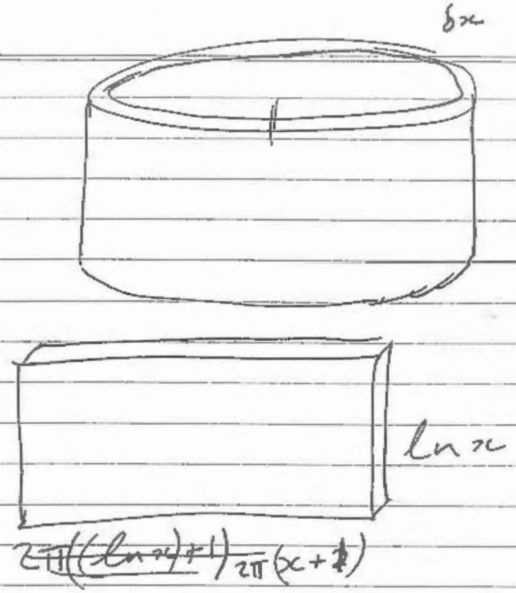
$\frac{\omega^7-1}{\omega-1}=0$

$\omega^7=1$

(iv) $\alpha=\frac{2\pi n}{7} \quad n=\pm1,\pm2,\pm3$

NOT $n=0$

(16)(a)



$\delta V = 2\pi(x+1)\ln x\,\delta x$ ✓

$V = 2\pi\int_1^e (x\ln x+\ln x)\,dx$

$v=\ln x \quad u'=x \qquad v=\ln x \quad u'=1$

$v'=\frac{1}{x} \quad u=\frac{x^2}{2} \qquad v'=\frac{1}{x} \quad u=x$

$=2\pi\left[\frac{x^2}{2}\ln x + x\ln x\right]_1^e - \int_1^e \frac{x}{2}+1\,dx$ ✓ ✓

$=2\pi\left(\left[\frac{e^2}{2}+e-0\right]-\left[\frac{x^2}{4}+x\right]_1^e\right)$

$=2\pi\left[\frac{e^2}{2}+e-\left(\frac{e^2}{4}+e-\frac{1}{4}-1\right)\right]$

$=2\pi\left(\frac{e^2}{4}+\frac{5}{4}\right)$ ✓

(i) $\Sigma F = 100 - kv^2$

$F = ma$

$100 - kv^2 = 20\ddot{x}$

$5 - \frac{kv^2}{20} = \ddot{x}$

(ii) $\ddot{x} = 5 - \frac{kv^2}{20}$

$v\frac{dv}{dx} = 5 - \frac{kv^2}{20}$

$\frac{dv}{dx} = \frac{5}{v} - \frac{kv}{20}$

$\frac{dv}{dx} = \frac{100 - kv^2}{20v}$

$\frac{dx}{dv} = \frac{20v}{100 - kv^2}$

$x = \frac{-10}{k}\int \frac{-2kv}{100 - kv^2}\,dv$

$= \frac{-10}{k}\ln(100 - kv^2) + C$

sub $x = 0$ $v = 0$

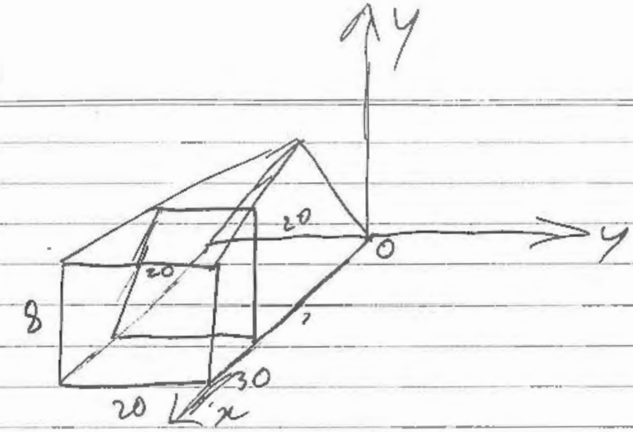
$C = \frac{10}{k}\ln 100$

$x = \frac{10}{k}\left(\ln 100 - \ln(100 - kv^2)\right)$

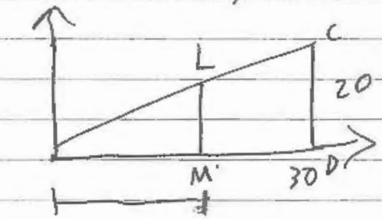
$= \frac{10}{k}\ln\left(\frac{100}{100 - kv^2}\right)$

16 c)

(i)



(ii) JK is clearly constant 20 cm



$m = \frac{20}{30} = \frac{2}{3}$

equation

$y = \frac{2}{3}x$

$ML = \frac{2}{3}x$

(iii) Area $= \frac{1}{2}\left(\frac{2}{3}x + 20\right) \times 8$

(iv) $V = \int_0^{30} \frac{1}{2}\left(\frac{2}{3}x + 20\right) \times 8\,dx$

$= 4\int_0^{30} \frac{2}{3}x + 20\,dx$

$= 4\left[\frac{x^2}{3} + 20x\right]_0^{30}$

$= 4(300 + 600)$

$= 3600\,u^3$